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The mathematics of fountain design: a multiple-centres activity

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Teachers of mathematics recognize the difficulty of reaching every student when the range of student abilities puts a considerable strain on the classroom discussion and time. In a response to the problem, students are grouped so that those with greater mathematical aptitude help those who have difficulties. While this approach is to be appreciated, it tends to mean that the more able students have less opportunity to explore further their own initiatives in mathematics, while those who have more difficulties find themselves on the receiving end with little opportunity to be in the role of enriching the mathematics experience for everyone, including themselves. A 'multiple-centres' approach is designed to overcome these problems. In this variation of differentiated instruction, all students get the chance to engage the material from a vantage point and at a level they find interesting and challenging as a consequence of their selecting extensions of the teacher's initial focus problem. This article will present some findings of 11th year (roughly Fifth Form) average mathematics students at a US Independent School in transforming the standard quadratic equation to represent fountain parabolic trajectories, which was the teacher's focus problem, along with some multiple-centre investigations they chose. A further set of opportunities with commentaries providing additional centres for student inquiry are included.

1. Introduction

Teachers of mathematics recognize the difficulty of reaching every student when the range of student abilities puts a considerable strain on the classroom discussion and time. In a response to the problem, students are grouped so that those with greater mathematical aptitude help those who have difficulties. While this approach is to be appreciated, it tends to mean that the more able students have less opportunity to explore further their own initiatives in mathematics, while those who have more difficulties find themselves on the receiving end with little opportunity to be in the role of enriching the mathematics experience for everyone, including themselves.

A 'multiple-centres' approach is designed to overcome these problems. In this variation of differentiated instruction, the teacher introduces an area of investigation say quadratic equations, by raising a particular problem, and after students alone and together, and with the teacher's assistance, come to make sense of that initial focus problem, the students and teacher establish other problems that students get the chance to choose from as extensions of their quadratic-equation investigations—in effect, creating multiple centres. After some time, the teacher brings the multiple centres of interest together for individual and group presentations, which may or may not be exclusively mathematical as some may be more artistic, historical or technological in nature that are associated with or inspired by the initial focus problem. In this way, each student has the opportunity to gain the appreciation of their classmates inasmuch as everyone has something to offer as a consequence of their further multiple-centre inquiries.

2. Establishing the focus problem and its ideational off-spring

The teacher's introduction of the quadratic equation by a discussion of the parameters of a parabolic arc determined by the path of a projectile over time given an initial velocity and height above the ground that is shaped by the effect of Earth's gravity sets the stage for the focus problem. Students see the resultant equation, $y = -\frac{g}{2}t^2 + vt + k$ as the height reached given the effect of gravity, an initial velocity and an initial height.

The teacher then raises the question: is what we know sufficient to represent the parabolic trajectories in terms of associating heights with distances from the initial release point, such as the parabolic arcs in fountains, inasmuch as all that has been considered at present is the height of the projectile as a function of time, not distance?

To promote this investigation, there are many visual opportunities where students can appreciate fountain designs. For example, McGinley Plaza and Discovery Park fountain were inspired by the Fibonacci sequence (www.purdue.edu/dp/dptour/fibonacci.pdf). Additionally, Helaman Ferguson's Fibonacci Fountain connects the Fibonacci sequence and the golden ratio. Ferguson's sculpture is based on the function $f(x) = (\frac{1+\sqrt{5}}{2})^{1/x}$ with $x = 0$ locating the point where the sculpture has its maximum height, as this point would be where the water would theoretically shoot to infinity, while other values of x are chosen along the length in terms of successive Fibonacci ratios, determining the respective heights the water reaches. (www.maa.org/mathland/mathtrek_10_21_02.html). Yet, regardless of the mathematical or technological sophistication, from the common drinking fountains to the extraordinary fountains that surprise and delight us in special settings, we recognize the familiar shape. The water leaving the fountain surface projects upward reaches a maximum height and then returns, all the while following the path of a parabolic trajectory. See Figs 1 and 2.

Student small-group discussions soon uncovered the new context required rethinking their basic understanding of the parameters of the quadratic equation. A quadratic equation that would represent the parabolic paths of water would have to include the angle associated with the entry of the water into the air, and also be able to determine the horizontal distance traveled if the water is sure to stay within the bounds of the fountain. The class effort to determine the equations with these parameters sets the focus for the common investigation. And the context of creating fountain designs provides many opportunities for students of varying mathematical, artistic, scientific and technical interest and facility to work alone and in small groups. Moreover, the setting provides the opportunity to include conversations about STEM considerations, where science, technology, engineering and mathematics are brought together.

3. The focus investigation

As with engaging any complex problem, considerable effort is required to formulate the problem so that it is in workable form. Accordingly, it was this aspect that required the most consistent teacher

FIG. 1. In preparation. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

FIG. 2. Variations on an elegant curve. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

intervention. Whereas representing the height of a concave parabolic curve as a function of time can be represented quite readily, with the new context requiring including the angle the water would exit the surface and the horizontal distance traveled as critical variables in their quadratic equations, the degree of complexity was raised considerably. Some students thought the variable of time would have to be replaced so as to write an equation in two variables, horizontal distance and the associated height. Yet they also understood how essential the variable of time was—they would need to take it into account to represent both the effect of gravity and the distance the water would travel as a function of the initial-exit velocity.

Some students sought to simplify the complexity by creating a right triangle with an angle of elevation representing the exit angle without considering the effects of gravity, though having both the horizontal distance, x, and the vertical distance, vt. This was an excellent application of the habit of mind, 'make the problem simpler', which would come in handy on other occasions. (See left side of Fig. 3). Other students sought to replace the horizontal distance, x, in terms of the new context, by $vt\cos D$, and the vertical distance h by $vt\sin D$ (see right side of Fig. 3).

From those initial understandings, two quadratic equations began to take form. Drawing upon the earlier consideration of combining the gravitational effect in conjunction with the linear distance the projectile would travel for a given time t, some students combined the height the water would reach, represented as (tan A)x, where A is the assumed exit angle, in combination with the effect of gravity on the water as a function of time. Students working with this model began to consider how to eliminate t from their equation $h(x) = -\frac{g}{2}t^2 + (\tan A)x$. Their effort led them to realize that as $\cos A = \frac{x}{vt}$ they could solve for t in terms of x, resulting in $h(x) = \frac{-\frac{g}{2}}{(v\cos A)^2}x^2 + (\tan A)x$. The students who had represented the horizontal and vertical distances in terms of the exit angle and time, so that $x(t) = vt\cos D$ and $h(t) = vt\sin D$, adjusted the latter equation to include the effect of gravity as a function of time to $h(t) = vt\sin D - \frac{g}{2}t^2 = (v\sin D)t - \frac{g}{2}t^2$, understanding that D was a constant, and that $x(t)$ would provide the horizontal distance as a function of time. The first equation allowed students to determine directly the height of the parabolic fountain arc as a function of the horizontal distance from the point where the water exited the fountain surface and the chosen exit angle. The second approach required two parametric equations, one to determine the height as a function of time, and the other to determine the distance associated with that height, also as a function of time. It made sense to everyone that zero would be the value of k in the standard quadratic equation as it expressed that the parabola's trajectory began at the water's surface.

Now all students had equations that made sense to them, and yet had little or no idea what specific values for the exit angle, the horizontal distance determined by the entry point, or the time, would actually mean in terms of determining heights or more completely a fountain arc. Naturally everyone tried various parameter values for their equations and graphed them, not unlike children playing in water, and came to the realization that both equations were valuable as they provided different perspectives and could answer different questions. Class and homework time were given to their associated investigations. The following represent some of the investigations they pursued and findings they made.

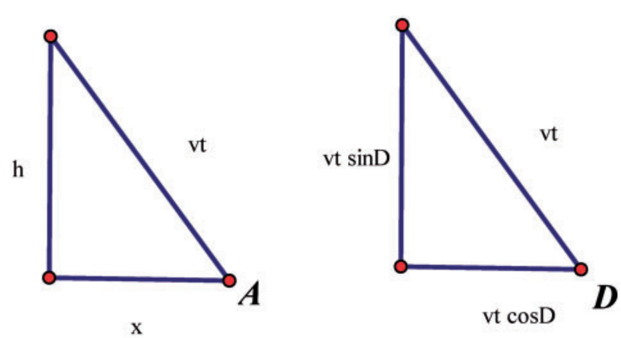

FIG. 3. Getting an angle on the problem. This figure appears in colour in the online version of *Teaching Mathematics and its Applications.*

4. Multiple-centres of investigation

(1) Students inspired by water jets wanted to find the heights of other vertical water fountain jet streams associated with various velocities. (Informing the students that this curve was known as a 'degenerate parabola' provided positive response.) Some Internet findings they made were as follows: King Fahd's fountain in Jeddah, Saudi Arabia, that reaches over 300 meters with water ejected at a speed of 390 km/h and the 130 m Jet d'Eau in Geneva, Switzerland, that had a speed of 210 km/h. The Bellagio Hotel fountain in Las Vegas, New Mexico, claimed that 64,600 l of water are in the air at the same time. Questions immediately began to surface: how much does the water in the air weigh? How much space does that much water take up? How long does the water stay in the air before returning to the surface? The more readily responded-to questions promoted a number of early experiences that essentially served to provide a sense of comfort and familiarity that made the consideration of more complex questions more available. It gave students who were less mathematically inclined more confidence that they could make headway and, it seemed, more resilience when things did get less clear. The questions raised that were more complex served to create multiple centres of inquiry for a number of students.

(2) Those students who had chosen the equation which included the tangent of the angle soon realized they could not determine the height of a vertical jet of water with their formula as $\tan(90^\circ)$ was undefined. However, those students with an equation in terms of time and the sine function were able to simplify their equation, for with $\sin(90^\circ) = 1$, they could consider vertical water jets using the equation $h(t) = vt - gt^2/2$. Some students drew upon their Internet data they had found to determine how long the water was in the air. Others realized that the maximum height would occur along the axis of symmetry, $x = -b/2a$, from their earlier study of the general equation, $y = ax^2 + bx + c$. Solving for t in their equation, they found that $t = -v/-g = v/g$, and so, $H(v/g) = v^2/g - (g/2)(v/g)^2 = v^2/2g$. Now they used the heights of the fountain jets data they had and determined the velocities to reach those heights, and checked their findings against those included in the Internet explorations. Knowing that $H = v^2/2g$ when $t = v/g$, they realized they could determine how long the water that reached the maximum height was actually in the air if they would restate v in terms of t. Doing the substitution some found that $t = 0.25(H^{1/2})$, and were quite pleased with their algebraic facility; then other students realized that this was the time in one direction, and needed to be doubled, so that the maximum time, T, in the air would be $T = 0.5(H^{1/2})$.

(3) The students, whose equation prevented them from considering vertical jets as the tangent is undefined at 90°, put their energies to finding the angle that would create the greatest width of a fountain for a given velocity. A few decided that moving in increments towards 0° from 90° would uncover the angle by trial and error (a time-honored habit of mind) or at least a range within which the most valuable angle would appear. A few students recalled the axis of symmetry was half the horizontal distance, so that doubling the horizontal distance would determine the point where the projected water would return to the surface. Using their equation, $h(x) = [-(g/2)/(v\cos A)^2]x^2 + (\tan A)x$, they determined the axis of symmetry, and after doubling, and doing some trigonometric substitution determined that the maximum distance, $X = 2(\tan A)(v\cos A)^2/g = 2v^2(\sin A)(\cos A)/g$. Now whatever the velocity, they sought to determine the angle A so to create the maximum distance, X. The calculator was given considerable attention, and after graphing $y = (\sin x)(\cos x)$ the consensus was that it would occur when $A = 45^\circ$, which confirmed their intuition. This led to concluding that the maximum distance would be when $X = v^2/g$. They then determined the maximum height at that angle to be $H(45^\circ) = v^2/4g$, and were pleased to discover that the ratio of maximum height to maximum width was 1:4.

(4) Some students knew their strengths were in drawing, while others understood they could best enrich the fountain-design experience with doing historical research. So, while they participated in all the mathematics discussions, they were more observers listening to the questions and discussions, coming to understand and appreciate the mathematical findings rather than generating them. This provided the opportunity for those students to make up equations after seeing others having done so and in the discussion process come to see the rationale for their own equations. Those who did historical research made a presentation on early fountain designs that educated everyone to the engineering feat of the extraordinary tiered aqueducts that brought water from the surrounding hills to the citizens of ancient Rome. They also greatly appreciated how fountain designers in the 17th century used gravity to create fountains that had water jetting into the air without any mechanical (or of course electrical) technology. They also learned that fountains were not just for visual pleasure, as most people gained their drinking water that way, and used the run off for washing. In comparison, fountains today may have pivoting nozzles controlled by computers that integrate lighting and music.

(5) Student artistry and mathematics also enriched the shared mathematics experience to a delightful degree (Fig. 4). In the figure that follows, the student's parabolic water arc #1 is to the right of water arc #2, with associated equations $y = [-16/(65\cos 75^\circ)^2](x+13)^2 + (\tan 75^\circ)(x+13)$ and $y = [-16/(65\cos 75^\circ)^2](x+25)^2 + (\tan 75^\circ)(x+25)$, where the velocities of both arcs are 65′/s, and the water leaving the dolphins' mouths are at an angle of 75°. (Note: these equations are in feet/s. as these students were more comfortable with this unit of measure.) The horizontal translation incorporated was appreciated by students along with the fact that the dolphins were to be 36 feet tall, while the pool was to have a diameter of 100 feet—quite a visual spectacle. Interestingly, the student chose to locate the graphic origin of the two-dimensional form at the

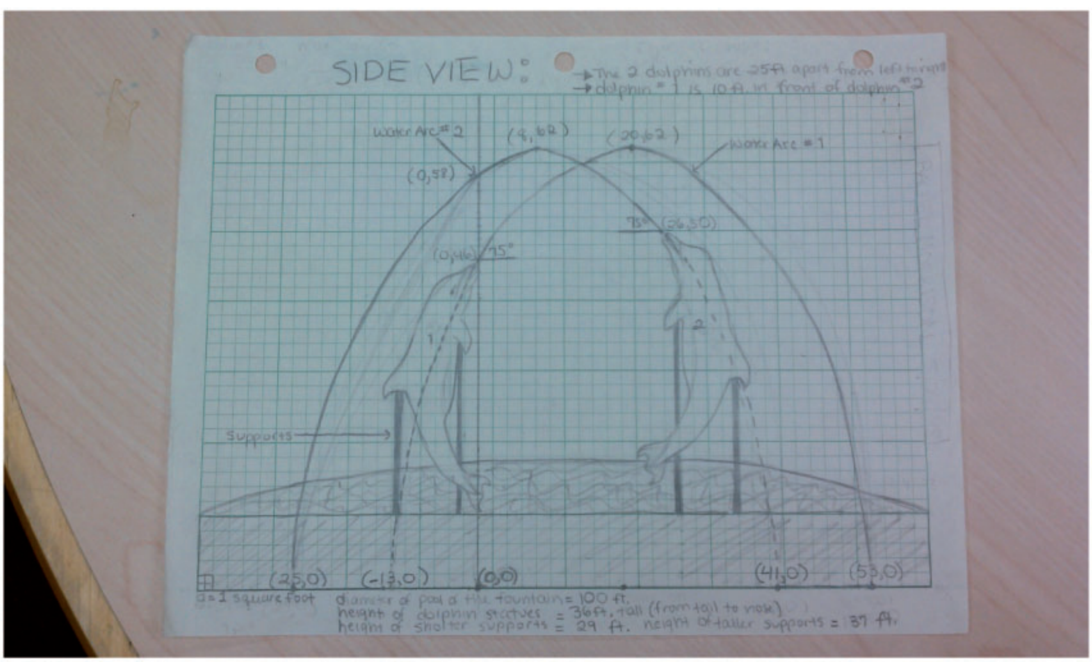

FIG. 4. A student's solution. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

base on the vertical below the nose of dolphin #1. That the dolphins do not appear to be reflected across an imagined vertical is due to the fact that the dolphin on the left is to be 10 feet in front of the dolphin on the right. A number of students came up with a formula for the volume of a cylinder, and the amount of water their fountain would hold. They were very impressed by the weight of the contained water, as a cubic foot weighed 62.4 lb (a cubic metre, 1680 lb). (And while they expressed interest in building one, complications of time and space were prohibitive.)

5. Centres for further student inquiry

There were other investigations students made. Beyond those just presented, the following are offered in that spirit.

(6) The physical apparatus to create a fountain using a hydraulic pump can be found at www.tryengineering.org/lessons/waterfountain.pdf.

(7) To determine flow rate assuming the fountain water exits a cylindrical pipe, the volume that passes each second would be determined by the product of the velocity of the water and area of the pipe surface. Students will need to experiment, and be careful of units, as pipe diameters are usually in centimetre for fountains.

(8) The weight of the water of a parabolic arc could be approximated by considering the arc as if it were a cylinder with a diameter equal to the diameter of the pipe. Based on that form, the volume could be determined, and as a result of multiplying by the density of water, an approximate weight of the water would be found.

(9) To determine the length of the parabolic curve without using calculus students can use their scale drawing and laying a string along the curve use that length adjusted to the original units. Some students may approximate half the length of the parabolic curve with the length of the hypotenuse of a right triangle where the base of the triangle would be half the distance from where the water entered the air and returned to the pool, while the height would be at the vertex (Fig. 5).

They will surely recognize that the approximation suffers from the hypotenuse being a straight segment. So encouraging them to think about how they might adjust for the additional length in their approximation formula might yield their tinkering (another valuable habit of mind) with the coefficient of the height variable, which provides them an experience engineers have. That is, they could consider different values of k in the expression for approximating the entire curve length, $2\sqrt{\left(\frac{b}{2}\right)^2 + k(h^2)}$, $k > 1$ and compare those values with what would be found measuring the parabolic arc lengths using the string. Civil engineer John Cresson Trautwine pointed out a century ago that setting $k = 4/3$ was 'the approximate rule given by various [civil engineering] pocket-books (1906, p. 192)'. Naturally, depending upon the magnitude of the curvature, the value is better or less of a good fit. For example, he noted that if height = base/4, the approximation was about ½% too much, but if height = 10 × base, then the error went to about 15½% too much.

(10) Determining the quadratic equations representing water arcs at school fountains will require careful measurements. Students will have to take into account that the point of entry is not at the same height as the endpoint in determining the equation representing the parabolic trajectory. Some students may discover that using the vertex can determine the angle measure inasmuch as $h = v^2(\sin A)^2/2g$ when $x = v^2(\sin A)(\cos A)/g$, and so $A = \tan^{-1}(2h/x)$.

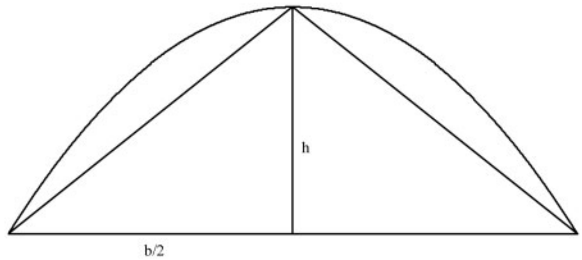

FIG. 5. Seeking a good approximation.

(11) Some students may be interested in determining how we actually get water coming out of sink faucets. They could learn how essential it is to make good decisions in locating reservoirs, and in laying miles of pipe, along with a realization of the practices and magnitude of the civil engineering involved. The economics involved could well be informative.
(12) Others might be interested in comparing a fountain's parabolic trajectory on Earth with one on the moon, given the latter's gravitational effect being approximately one-sixth that of the Earth's. Clearly, water exiting the fountain pool on the surface of the moon at the same velocity as on Earth will have a greater height and total flight time in the rarified air. They will find that the total flight time will be around 2½ times as long as on Earth.
(13) To appreciate the science of this technology, students might want to research laminar flow nozzles. These nozzles are constructed to remove turbulence, so the water arcs appear as if they are made of glass. There are many websites that demonstrate how to construct one.
(14) Students artistically inclined might appreciate that it has been said 'architecture is frozen music', and create some forms, or series of jet heights, that 'play out a tune'. Others might want to create parabolic trajectories that suggest semi-circular arcs, which would be in confluence with their circular fountain base—in effect serving as a rotation of the curve.
(15) More can be done of course in terms of uncovering physical relationships, but some assistance in terms of the physics involved may be necessary. For example, it can be shown that the relationship between height in feet and pressure, P, in terms of psi, is: $h = 2.31P$; and too, that volumetric flow rate $= 30d^2\sqrt{P}$, where d is the diameter of the water pipe and P is the pressure.
(16) Those students who appreciate technology could consider radar and television 'dishes', and uncover the rationale for their having parabolic shape.
(17) Some students whose interest is of a more abstract nature may wish to look into the mathematical origins of the parabola. The ancient Greeks who shared that interest were evidently compelled—after having determined they could define a circle as a set of points at a given distance from a given point, they investigated what form would be created by a set of points at a given distance from a given point and a given line.

6. Conclusion

A multiple-centres inquiry is offered so that every mathematics student will find educational opportunities to enrich their own and others' awareness and understanding, and be appreciated for doing so. The openness of the inquiries tends to promote students working things out alone and together, and drawing upon mathematical habits of mind as they do not readily have algorithms or models to apply. Both aspects would suggest the activity will invariably serve them well with regard to solving problems in many contexts. And if we want to promote a society where collaboration is valued, then educational institutions need to support that activity, despite the ‘time wasted’ relative to the efficiency of a teacher lecture. It would seem the more such experiences students have for seeing how mathematics can provide insights into the world we inhabit and create, the more they are able to appreciate both the thinking of the ancient Pythagoreans and that of modern-day engineers, as well as their own.

Reference

Trautwine, J. (1906) *The Civil Engineer’s Pocket-Book*, New York: Wiley.

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